On the Validity of Meteor Radiants deduced from Three Tracks.

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1. It is not uncommon in tables of radiants of meteors to find radiants deduced from very few observations: four is quite a common number, and three by no means rare. We see, however, that it is quite possible that three or four paths, although quite unconnected, should happen to pass so closely through the same point that they would be taken to indicate a radiant there. It therefore becomes of interest to determine the probability of this happening. Suppose the mean divergence of an observed path from the radiant is ω, then it is clear that if the inradius of the Δ formed by the paths of three meteors $\ngtr \omega$ the three paths will be considered to pass through the same point and, subject to certain conditions which will be investigated later, to indicate a radiant there.

The problem before us is, therefore, to find the chance that the incircle of the Δ formed by three great circles taken at random on a sphere $\ngtr \omega$.

2. A uniform distribution of great circles on a sphere corresponds to a uniform distribution of their poles; accordingly we must have the circles so placed that it is possible to find a pole of each forming a Δ of circumradius $> \frac{\pi}{2} - \omega$.

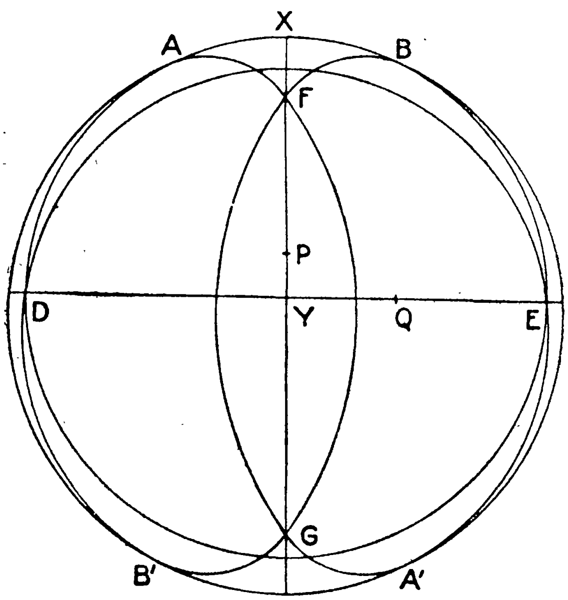

Fig. 1.

Let A (fig. 1) be a pole of one of the circles, and B that pole of the second which is nearer to A.

The chance of AB falling between x and $x+dx$

$$=\frac{2\pi \sin x dx}{2\pi} \sin x dx.$$

Then if $x < 2\omega$ any $\odot$ through A and B′, the point antipodal to B has a radius $> \frac{1}{2}\pi - \omega$, and the conditions are certainly satisfied. If $x > 2\omega$, let A′ be the point antipodal to A, and through A, B; A, B′; A′, B; and A′, B′ draw circles of radius $\frac{1}{2}\pi - \omega$. We will for the present only consider the four of the eight circles whose major arcs lie on one of the two hemispheres into which the sphere is divided by the plane ABA′B′.

3. We will now find the condition that the circles through A, B; A′, B′ may cut each other. Let P be the centre of the $\odot$ through AB, and draw PX perp. to AB. Then

$$\cos PX = \cos AP \sec AX = \sin \omega \sec \tfrac{1}{2}x,$$

and we see that the circles will cut if

$$2\{PX + (\tfrac{1}{2}\pi - \omega)\} > \pi,$$

i.e. $$PX > \omega,$$

i.e. $$\sin \omega \sec \tfrac{1}{2}x < \cos \omega,$$

i.e. $$\cos \tfrac{1}{2}x > \tan \omega.$$

Since $x < \frac{\pi}{2}$, this is always true if $\omega < \tan^{-1}\frac{1}{\sqrt{2}}$, which is the case, ω being small.

∴ the circles through A, B; A′, B′ will cut; let them cut in D, E (fig. 2).

Similarly the circles through A, B′; A′, B will cut if

$$\sin\frac{x}{2} > \tan \omega,$$

i.e. $$x > 2 \sin^{-1} \tan \omega.$$

Now it will be seen that the conditions will not be satisfied if C, the pole of the third circle which lies in the hemisphere considered, fall within all four of the circles through A, B; A, B′; A′, B; A′, B′, but will in every other case.

The conditions will be satisfied if C falls within both of the circles through any one of our four pairs of points; but this gives nothing new, for the second circle through AB is antipodal to A′B′D and so can have no points in common with it, so that our conditions are already satisfied by C being outside A′B′D, and similarly for the other pairs of points.

Now if the circles through A′, B; A, B′ do not cut, the four

circles will have no common portion and the conditions will be satisfied. Now these circles will not cut unless $x > 2\sin^{-1}\tan\omega$,

$$\therefore \text{ the chance in this way} = \int_0^{2\sin^{-1}\tan\omega} \sin x\, dx$$

$$= 1 - \cos 2(\sin^{-1}\tan\omega)$$

$$= 2\tan^2\omega \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

4. Consider now the case when the circles through A, B′; A′, B cut in F, G.

Then we have two subcases, according as the points F, G lie both inside or both outside of the area common to ABD, A′B′D.

X, P, F, G will clearly lie on the same great $\odot$.

Then F and G will both lie inside the area common to ABD, ABD′ if

$$\mathrm{FG} < 2\{\tfrac{1}{2}(\pi - \omega + \mathrm{PX}) - \tfrac{1}{2}\pi\}$$
$$< 2(\mathrm{PX} - \omega).$$

To find FG, let Q be the centre of A′BG and draw QY perp. to FG. Then $\cos\frac{1}{2}\mathrm{FG} = \cos\mathrm{QF}\sec\mathrm{QY}$.

Let QY cut AB A′B′ in Z.

$$\begin{aligned}
\text{Then} \qquad \mathrm{QY} &= \tfrac{1}{2}\pi - \mathrm{QZ} \\
&= \tfrac{1}{2}\pi - \cos^{-1}(\cos\mathrm{QB}\sec\mathrm{BZ}) \\
&= \tfrac{1}{2}\pi - \cos^{-1}(\sin\omega\,\mathrm{cosec}\,\tfrac{1}{2}x) \\
\text{and} \qquad \mathrm{QF} &= \tfrac{1}{2}\pi - \omega\,;
\end{aligned}$$

$$\begin{aligned}
\therefore \cos\tfrac{1}{2}\mathrm{FG} &= \sin\omega\,\mathrm{cosec}\cos^{-1}(\sin\omega\,\mathrm{cosec}\,\tfrac{1}{2}x) \\
&= \sin\omega\sin\tfrac{1}{2}x\,(\sin^2\tfrac{1}{2}x - \sin^2\omega)^{-\frac{1}{2}}.
\end{aligned}$$

No ambiguity exists as to the sign of the root, as $\frac{1}{2}\mathrm{FG} < \frac{1}{2}\pi$; $\therefore$ our position becomes

$$\cos^{-1}\{\sin\omega\sin\tfrac{1}{2}x\,(\sin^2\tfrac{1}{2}x - \sin^2\omega)^{-\frac{1}{2}}\} < \cos^{-1}(\sin\omega\sec\tfrac{1}{2}x) - \omega,$$

i.e.
$$\sin\omega\sin\tfrac{1}{2}x(\sin^2\tfrac{1}{2}x - \sin^2\omega)^{-\frac{1}{2}} > \sin\omega\cos\omega\sec\tfrac{1}{2}x + \sin\omega\sec\tfrac{1}{2}x\,(\cos^2\tfrac{1}{2}x - \sin^2\omega)^{\frac{1}{2}}.$$

This on reduction gives

$$4\sin^4\tfrac{1}{2}x - \sin^2\tfrac{1}{2}x\,(8 - 5\cos^2\omega) + 4\sin^2\omega > 0,$$

i.e.
$$\sin^2\tfrac{1}{2}x < \tfrac{1}{8}\{8 - 5\cos^2\omega - \cos\omega\,(25\cos^2\omega - 16)^{\frac{1}{2}}\}$$
or
$$> \tfrac{1}{8}\{8 - 5\cos^2\omega + \cos\omega\,(25\cos^2\omega - 16)^{\frac{1}{2}}\}.$$

The second of these alternatives is impossible, for $\sin^2\frac{1}{2}x \not> \frac{1}{2}$, and if we put

$$\tfrac{1}{8}\{8 - 5\cos^2\omega + \cos\omega\,(25\cos^2\omega - 16)^{\frac{1}{2}}\} = \tfrac{1}{2}$$

we get on solving $\qquad \cos^2\omega = \frac{2}{3}$;

and $\cos\omega$ must be positive, for $0 < \omega < \frac{1}{2}\pi$.

But this will not satisfy our equation; it satisfies

$$\tfrac{1}{8}\{8-5\cos^2\omega-\cos\omega\,(25\cos^2\omega-16)^{\frac{1}{2}}\}=\tfrac{1}{2}.$$

So $\frac{1}{8}\{8-5\cos^2\omega+\cos\omega\,(25\cos^2\omega-16)^{\frac{1}{2}}\}$ never $=\frac{1}{2}$, and it is sometimes greater, so it is always greater, since it is continuous.

So the second condition is never fulfilled.

The other condition is

$$x<2\sin^{-1}\tfrac{1}{8}\{8-5\cos^2\omega-\cos\omega\,(25\cos^2\omega-16)^{\frac{1}{2}}\}^{\frac{1}{2}}<\theta \quad \ldots \quad (2)$$

This assumes $25\cos^2\omega-16\not<0$,

$\therefore$ $\cos\omega\not<\frac{4}{5}$,

i.e. $\omega<36°$ about, which will be true.

For this subcase to actually occur we must have

$$\theta>2\sin^{-1}\tan\omega.$$

We can see that this will always happen by considering fig. 1, for what we assert is that there are always real positions of F and G inside the figure DE; and we can see that this is true, for ABD, A′B′D always cut, so that DE always exists, and is of finite size, even when F and G coincide in Y, so that F and G are inside the figure DE in this case, and cannot get outside it until FG has attained a certain finite size.

5. We will now consider the subcase when $2\sin^{-1}\tan\omega<x<\theta$ and F and G lie inside DE, fig. 1. Then the conditions will be fulfilled if C lies outside the figure FG, the chance of which is

$$\frac{2\pi-\text{area FG}}{2\pi}.$$

And area FG $= 2(\text{sector FQG}-\Delta\text{FQG})$

$$= 2\{(1-\sin\omega)\text{F}\hat{\text{Q}}\text{G}+(\pi-\text{F}\hat{\text{Q}}\text{G}-2\text{F}\hat{\text{G}}\text{Q})\}$$

$$= 2\pi-2\sin\omega\,\text{F}\hat{\text{Q}}\text{G}-4\text{F}\hat{\text{G}}\text{Q};$$

$\therefore$ chance of conditions being fulfilled $=\pi^{-1}(\sin\omega\,\text{F}\hat{\text{Q}}\text{G}+2\text{F}\hat{\text{G}}\text{Q})$.

$$\text{F}\hat{\text{Q}}\text{G}=2\text{Y}\hat{\text{Q}}\text{F}$$

$$=2\cos^{-1}(\tan \text{QY}\cot \text{QF})$$

$$=2\cos^{-1}\{\tan\omega\cot\cos^{-1}(\sin\omega\operatorname{cosec}\tfrac{1}{2}x)\}$$

$$=2\cos^{-1}\left\{\sin\omega\tan\omega\left(\sin^2\frac{x}{2}-\sin^2\omega\right)^{-\frac{1}{2}}\right\}$$

and $\text{F}\hat{\text{G}}\text{Q}=\sin^{-1}(\sin \text{QY}\operatorname{cosec}\text{QF})$

$$=\sin^{-1}\left(\tan\omega\operatorname{cosec}\frac{x}{2}\right);$$

$\therefore$ chance of conditions being fulfilled

$$=2\pi^{-1}\left[\sin\omega\cos^{-1}\left\{\sin\omega\tan\omega\left(\sin^2\frac{x}{2}-\sin^2\omega\right)^{-\frac{1}{2}}\right\}+\sin^{-1}\left(\tan\omega\operatorname{cosec}\frac{x}{2}\right)\right];$$

∴ the total chance of conditions being fulfilled under this subcase

$$=\int_{2\sin^{-1}\tan\omega}^{\theta} 2\pi^{-1}\left[\sin\omega\cos^{-1}\left\{\sin\omega\tan\omega\left(\sin^2\frac{x}{2}-\sin^2\omega\right)^{-\frac{1}{2}}\right\}\right.$$
$$\left.+\sin^{-1}\left(\tan\omega\operatorname{cosec}\frac{x}{2}\right)\right]\sin x\,dx.$$

Now $\int\cos^{-1}\{\sin\omega\tan\omega(\sin^2\tfrac{1}{2}x-\sin^2\omega)^{-\frac{1}{2}}\}\sin x\,dx$

$$= -\cos x\cos^{-1}\{\sin\omega\tan\omega\,(\sin^2\tfrac{1}{2}x-\sin^2\,\omega^{-\frac{1}{2}}\}$$
$$+\tfrac{1}{2}\sin\omega\tan\omega\int\cos x\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x\,(\sin^2\tfrac{1}{2}x-\sin^2\omega)^{-1}$$
$$(\sin^2\tfrac{1}{2}x-\tan^2\omega)^{-\frac{1}{2}}\,dx$$

$$= -\cos x\cos^{-1}\{\sin\omega\tan\omega\,(\sin^2\tfrac{1}{2}x-\sin^2\omega)^{-\frac{1}{2}}\}$$
$$+2^{-\frac{1}{2}}\sin\omega\tan\omega\int\sin x\cos x\,(\cos 2\omega-\cos x)^{-1}$$
$$(1-2\tan^2\omega-\cos x)^{-\frac{1}{2}}\,dx$$

$$\int\sin x\cos x\,(\cos 2\omega-\cos x)^{-1}(1-2\tan^2\omega-\cos x)^{-\frac{1}{2}}\,dx$$
$$= -\int\sin x\,(1-2\tan^2\omega-\cos x)^{-\frac{1}{2}}\,dx$$
$$+\cos 2\omega\int\sin x\,(\cos 2\omega-\cos x)^{-1}(1-2\tan^2\omega-\cos x)^{-\frac{1}{2}}\,dx$$
$$= -2(1-2\tan^2\omega-\cos x)^{\frac{1}{2}}$$
$$+2\cos 2\omega\int(\cos 2\omega-1+2\tan^2\omega+z^2)^{-1}\,dz,$$

putting $1-2\tan^2\omega-\cos x=z^2$

$$= -2(1-2\tan^2\omega-\cos x)^{\frac{1}{2}}$$
$$+2^{\frac{1}{2}}\operatorname{cosec}\omega\cot\omega\cos 2\omega\tan^{-1}(2^{-\frac{1}{2}}\operatorname{cosec}\omega\cot\omega\,z)$$
$$= -2(1-2\tan^2\omega-\cos x)^{\frac{1}{2}}+2^{\frac{1}{2}}\cos 2\omega\operatorname{cosec}\omega\cot\omega$$
$$\tan^{-1}\{\operatorname{cosec}\omega\cot\omega\,2^{-\frac{1}{2}}(1-2\tan^2\omega-\cos x)^{\frac{1}{2}}\}$$

$$\therefore\int\cos^{-1}\{\sin\omega\tan\omega\,(\sin^2\tfrac{1}{2}x-\sin^2\omega)^{-\frac{1}{2}}\}\sin x\,dx$$
$$= -\cos x\cos^{-1}\{\sin\omega\tan\omega\,(\sin^2\tfrac{1}{2}x\sin^2\omega)^{-\frac{1}{2}}\}$$
$$-\sin\omega\tan\omega\,2^{\frac{1}{2}}\,(1-2\tan^2\omega-\cos x)^{\frac{1}{2}}+\cos 2\omega\tan^{-1}$$
$$\{\operatorname{cosec}\omega\cot\omega\,2^{-\frac{1}{2}}\,(1-2\tan^2\omega-\cos x)^{\frac{1}{2}}\};$$

$$\therefore\int_{2\sin^{-1}\tan\omega}^{\theta}2\sin\omega\,\pi^{-1}\cos^{-1}\{\sin\omega\tan\omega\left(\sin^2\frac{x}{2}-\tan^2\omega\right)^{-\frac{1}{2}}\}\sin x\,dx$$
$$= 2\sin\omega\,\pi^{-1}[-\cos\theta\cos^{-1}\{\sin\omega\tan\omega\,(\sin^2\tfrac{1}{2}\theta-\tan^2\omega)^{-\frac{1}{2}}\}$$
$$-\sin\omega\tan\omega\,2^{\frac{1}{2}}\,(1-2\tan^2\omega-\cos\theta)^{\frac{1}{2}}+\cos 2\omega\tan^{-1}$$
$$\{2^{-\frac{1}{2}}\operatorname{cosec}\omega\cot\omega\,(1-2\tan^2\omega-\cos\theta)^{\frac{1}{2}}\}]$$

and

$$\int \sin^{-1}(\tan\omega \operatorname{cosec}\tfrac{1}{2}x)\sin x\,dx = -\cos x \sin^{-1}(\tan\omega\operatorname{cosec}\tfrac{1}{2}x)$$
$$-\int \tfrac{1}{2}\tan\omega\cos x\cos\tfrac{1}{2}x\operatorname{cosec}^2\tfrac{1}{2}x\,(1-\tan^2\omega\operatorname{cosec}^2\tfrac{1}{2}x)^{-\frac{1}{2}}\,dx$$

$$\int \tfrac{1}{2}\tan\omega\cos x\cos\tfrac{1}{2}x\operatorname{cosec}^2\tfrac{1}{2}x\,(1-\tan^2\omega\operatorname{cosec}^2\tfrac{1}{2}x)^{-\frac{1}{2}}\,dx$$
$$=\int \tan\omega\,(\tfrac{1}{2}\cos\tfrac{1}{2}x\operatorname{cosec}^2\tfrac{1}{2}x-\cos\tfrac{1}{2}x)(1-\tan^2\omega\operatorname{cosec}^2\tfrac{1}{2}x)^{-\frac{1}{2}}\,dx$$
$$=-\sin^{-1}(\tan\omega\operatorname{cosec}\tfrac{1}{2}x)-\int\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x\tan\omega\,(\sin^2\tfrac{1}{2}x-\tan^2\omega)^{-\frac{1}{2}}\,dx$$
$$=-\sin^{-1}(\tan\omega\operatorname{cosec}\tfrac{1}{2}x)-2\tan\omega\,(\sin^2\tfrac{1}{2}x-\tan^2\omega)^{\frac{1}{2}};$$
$$\therefore \int\sin^{-1}(\tan\omega\operatorname{cosec}\tfrac{1}{2}x)\sin x\,dx = 2\sin^2\tfrac{1}{2}x\sin^{-1}(\tan\omega\operatorname{cosec}\tfrac{1}{2}x)$$
$$+2\tan\omega\,(\sin^2\tfrac{1}{2}x-\tan^2\omega)^{\frac{1}{2}};$$

$$\therefore \int_{2\sin^{-1}\tan\omega}^{\theta} 2\pi^{-1}\sin^{-1}(\tan\omega\operatorname{cosec}\tfrac{1}{2}x)\sin x\,dx = 4\pi^{-1}\sin^2\tfrac{1}{2}\theta\sin^{-1}(\tan\omega\operatorname{cosec}\tfrac{1}{2}\theta)$$
$$+4\pi^{-1}\tan\omega\left(\sin^2\frac{\theta}{2}-\tan^2\omega\right)^{\frac{1}{2}}-2\tan^2\omega;$$

$\therefore$ total chance under this subcase is

$$2\pi^{-1}\sin\omega\,[\cos 2\omega\tan^{-1}\{2^{-\frac{1}{2}}\operatorname{cosec}\omega\cot\omega\,(1-2\tan^2\omega-\cos\theta)^{\frac{1}{2}}\}$$
$$-\cos\theta\cos^{-1}\{\sin\omega\tan\omega\,(\sin^2\tfrac{1}{2}\theta-\sin^2\omega)^{-\frac{1}{2}}\}$$
$$-\sin\omega\tan\omega\,2^{\frac{1}{2}}(1-2\tan^2\omega-\cos\theta)^{\frac{1}{2}}]+4\pi^{-1}\sin^2\tfrac{1}{2}\theta\sin^{-1}(\tan\omega\operatorname{cosec}\tfrac{1}{2}\theta)$$
$$+4\pi^{-1}\tan\omega\,(\sin^2\tfrac{1}{2}\theta-\tan^2\omega)^{\frac{1}{2}}-2\tan^2\omega \quad \text{. . . . .} \quad (3)$$

6. If F and G lie outside the area common to ABD, A′B′D, (fig. 2), let ABD cut AFB′ in H and A′FB in K, and let A′B′D cut AFB′ in L and A′FB in M. Then the conditions will be satisfied if C falls outside HKML, and then only.

Now AKHB + B′HLA + B′MLA′ + A′KMB
= AFB + AFMD + MDK + FML + BFLE + LEH + HKML
+ ADB′ + DKGB′ + HKG + MDK + AFMD + FML + HKML
+ A′GB′ + GHEA′ + LEH + HKG + B′DKG + MDK + HKML
+ A′EB + FLEB + FML + LEH + A′GHE + GHK + HKML
= 2π + FML + BFLE + LEH + HKML
+ MDK + AFMD + FML + HKML
+ HKG + B′DKG + MDK + HKML
+ LEH + A′GHE + GHK + HKML − HKML
= 2π + 4BFKE − HKML;

$\therefore$ HKML = 2π + 4BFKE − 2AKHB + 2AHB′;

∴ chance of conditions being fulfilled

$$= \frac{2\pi - \text{HKML}}{2\pi} = \pi^{-1}(\text{AKHB} + \text{AFHB}' - 2\text{BFKE}).$$

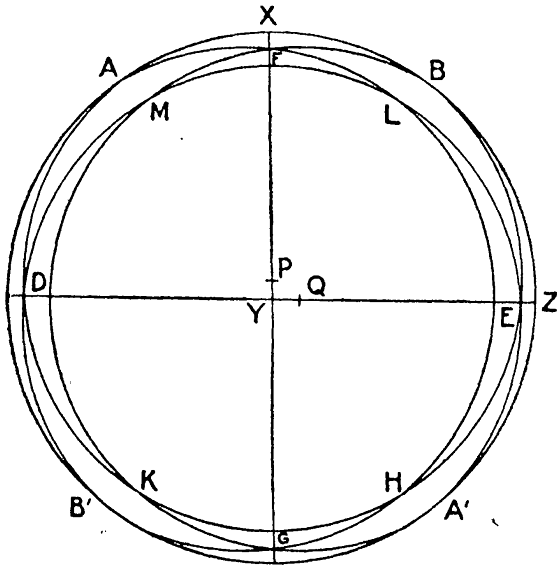

Fig. 2.

7. Now

$$\text{P}\hat{\text{B}}\text{A} = \cos^{-1}(\tan \tfrac{1}{2}x \tan \omega)$$
$$\text{QBA}' = \cos^{-1}(\cot \tfrac{1}{2}x \tan \omega);$$
$$\therefore\ \text{P}\hat{\text{B}}\text{Q} = \pi - \cos^{-1}(\tan \tfrac{1}{2}x \tan \omega) - \cos^{-1}(\cot \tfrac{1}{2}x \tan \omega)$$
$$= \sin^{-1}(\tan \tfrac{1}{2}x \tan \omega) + \sin^{-1}(\cot \tfrac{1}{2}x \tan \omega),$$

and

$$\begin{aligned}\text{P}\hat{\text{B}}\text{K} &= \text{K}\hat{\text{B}}\text{M} - \text{P}\hat{\text{B}}\text{M} \\ &= \tfrac{1}{2}\text{H}\hat{\text{B}}\text{M} - (\tfrac{1}{2}\pi - \text{P}\hat{\text{B}}\text{Q}) \\ &= \tfrac{1}{2}(\text{Q}\hat{\text{B}}\text{M} + \text{P}\hat{\text{B}}\text{M} - \text{P}\hat{\text{B}}\text{Q}) - (\tfrac{1}{2}\pi - \text{P}\hat{\text{B}}\text{Q}) \\ &= \tfrac{1}{2}\text{P}\hat{\text{B}}\text{Q} \\ &= \tfrac{1}{2}\{\sin^{-1}(\tan \tfrac{1}{2}x \tan \omega) + \sin^{-1}(\cot \tfrac{1}{2}x \tan \omega)\}.\end{aligned}$$

Also

$$\cot \text{B}\hat{\text{P}}\text{Q} = \tan \text{P}\hat{\text{B}}\text{K} \cos \text{PB} = \tan \text{P}\hat{\text{B}}\text{K} \sin \omega$$

and area

$$\begin{aligned}\text{BFKE} &= 2 \text{ sector PBK} - 2\Delta\, \text{PBK} \\ &= 2(1 - \sin \omega) \times 2\text{B}\hat{\text{P}}\text{Q} - 2(2\text{B}\hat{\text{P}}\text{Q} + 2\text{P}\hat{\text{B}}\text{K} - \pi) \\ &= 2\pi - 4\text{P}\hat{\text{B}}\text{K} - 4 \sin \omega \text{B}\hat{\text{P}}\text{Q} \\ &= 2\pi - 2\sin^{-1}(\tan \tfrac{1}{2}x \tan \omega) - 2\sin^{-1}(\cot \tfrac{1}{2}x \tan \omega) \\ &\quad - 4 \sin \omega \cot^{-1}[\sin \omega \tan \tfrac{1}{2}\{\sin^{-1}(\cot \tfrac{1}{2}x \tan \omega) \\ &\quad + \sin^{-1}(\tan \tfrac{1}{2}x \tan \omega)\}].\end{aligned}$$

Also

$$\begin{aligned}\text{AKHB} &= \text{sector PAKHB} + \Delta\text{APB}\\ &= (1-\sin\omega)(2\pi - 2x\hat{\text{P}}\text{B}) + 2x\hat{\text{P}}\text{B} + 2\text{A}\hat{\text{B}}\text{P} - \pi\\ &= \pi(1-2\sin\omega) + 2\sin\omega\sin^{-1}(\sin\tfrac{1}{2}x\sec\omega)\\ &\qquad + 2\cos^{1}(\tan\tfrac{1}{2}x\tan\omega)\end{aligned}$$

Similarly

$$\begin{aligned}\text{AFHB}' &= \pi(1-2\sin\omega) + 2\sin\omega\sin^{-1}(\cos\tfrac{1}{2}x\sec\omega)\\ &\qquad + 2\cos^{-1}(\cot\tfrac{1}{2}x\tan\omega);\end{aligned}$$

$\therefore$ $\quad$ AKHB + AFAB′ − 2BFKE

$$\begin{aligned}&= -2\pi(1+2\sin\omega) + 2\sin\omega\sin^{-1}(\sin\tfrac{1}{2}x\sec\omega)\\ &\qquad + 2\cos^{-1}(\tan\tfrac{1}{2}x\tan\omega) + 2\sin\omega\sin^{-1}(\cos\tfrac{1}{2}x\sec\omega)\\ &\qquad + 2\cos^{-1}(\cot\tfrac{1}{2}x\tan\omega)\\ &\qquad + 8\sin\omega\cot^{-1}[\sin\omega\tan\tfrac{1}{2}\{\sin^{-1}(\cot\tfrac{1}{2}x\tan\omega)\\ &\qquad + \sin^{-1}(\tan\tfrac{1}{2}x\tan\omega)\}]\\ &= 2\sin\omega\sin^{-1}(\sin\tfrac{1}{2}x\sec\omega) + 2\sin\omega\sin^{-1}(\cos\tfrac{1}{2}x\sec\omega)\\ &\qquad + 2\sin^{-1}(\tan\tfrac{1}{2}x\tan\omega) + 2\sin^{-1}(\cot\tfrac{1}{2}x\tan\omega)\\ &\qquad - 8\sin\omega\tan^{-1}[\sin\omega\tan\tfrac{1}{2}\{\sin^{-1}(\cot\tfrac{1}{2}x\tan\omega)\\ &\qquad + \sin^{-1}(\tan\tfrac{1}{2}x\tan\omega)\}];\end{aligned}$$

$\therefore$ total chance of conditions being fulfilled under this subcase

$$\begin{aligned}= 2\pi^{-1}\int_{\theta}^{\frac{\pi}{2}}\Big\{&\sin\omega\sin^{-1}(\sin\tfrac{1}{2}x\sec\omega) + \sin\omega\sin^{-1}(\cos\tfrac{1}{2}x\sec\omega)\\ &+ \sin^{-1}(\tan\tfrac{1}{2}x\tan\omega) + \sin^{-1}(\tan\tfrac{1}{2}x\tan\omega)\\ &- 4\sin\omega\tan^{-1}[\sin\omega\tan\tfrac{1}{2}\{\sin^{-1}(\cot\tfrac{1}{2}x\tan\omega)\\ &+ \sin^{-1}(\tan\tfrac{1}{2}x\tan\omega)\}]\Big\}\sin x\,dx \qquad \ldots \quad (4)\end{aligned}$$

8. $\displaystyle\int\sin^{-1}(\cot\tfrac{1}{2}x\tan\omega)\sin x\,dx$

$$\begin{aligned}&= -\cos x\sin^{-1}(\cot\tfrac{1}{2}x\tan\omega)\\ &\quad - \int\tfrac{1}{2}\cos x\tan\omega\,\mathrm{cosec}^2\tfrac{1}{2}x(1-\cot^2\tfrac{1}{2}x\tan^2\omega)^{-\frac{1}{2}}dx\\ &= -\cos x\sin^{-1}(\cot\tfrac{1}{2}x\tan\omega)\\ &\quad - \int\tfrac{1}{2}\tan\omega\cos x\,\mathrm{cosec}\,\tfrac{1}{2}x(\sin^2\tfrac{1}{2}x - \tan^2\omega\cos^2\tfrac{1}{2}x)^{-\frac{1}{2}}dx\end{aligned}$$

$$\begin{aligned}\int\tfrac{1}{2}\tan\omega\cos x\,\mathrm{cosec}\,\tfrac{1}{2}x&(\sin^2\tfrac{1}{2}x - \tan^2\omega\cos^2\tfrac{1}{2}x)^{-\frac{1}{2}}dx\\ &= \int\tfrac{1}{2}\tan\omega\,\mathrm{cosec}\,\tfrac{1}{2}x(1-\sec^2\omega\cos^2\tfrac{1}{2}x)^{-\frac{1}{2}}dx\\ &\quad - \int\tan\omega\sin\tfrac{1}{2}x(1-\sec^2\omega\cos^2\tfrac{1}{2}x)^{-\frac{1}{2}}dx\\ &= \int\tfrac{1}{2}\tan\omega\,\mathrm{cosec}\,\tfrac{1}{2}x(1-\sec^2\omega\cos^2\tfrac{1}{2}x)^{-\frac{1}{2}}dx\\ &\qquad + 2\sin\omega\sin^{-1}(\cos\tfrac{1}{2}x\sec\omega)\\ &= 2\sin\omega\sin^{-1}(\cos\tfrac{1}{2}x\sec\omega) + \int\tan\omega(u^2+\tan^2\omega)^{-1}du;\end{aligned}$$

putting

$$1-\cos^2\tfrac{1}{2}x\sec^2\tfrac{1}{2}\omega=u^2\cos^2\tfrac{1}{2}x$$

$$=2\sin\omega\sin^{-1}(\cos\tfrac{1}{2}x\sec\omega)+\tan^{-1}(u\cot\omega)$$

$$=2\sin\omega\sin^{-1}(\cos\tfrac{1}{2}x\sec\omega)+\tan^{-1}\{\cot\omega(\sec^2\tfrac{1}{2}x-\sec^2\omega)^{-\frac{1}{2}}\};$$

$$\therefore\int\sin^{-1}(\cot\tfrac{1}{2}x\tan\omega)\sin x\,dx$$

$$=-\cos x\sin^{-1}(\cot\tfrac{1}{2}x\tan\omega)-2\sin\omega\sin^{-1}(\cos\tfrac{1}{2}x\sec\omega)-\tan^{-1}\{\cot\omega(\sec^2\tfrac{1}{2}x-\sec^2\omega)^{-\frac{1}{2}}\}$$

and $$\int_\theta^{\frac{\pi}{2}}\sin^{-1}(\tan\tfrac{1}{2}x\tan\omega)\sin x\,dx$$

$$=-\int_{\pi-\theta}^{\frac{\pi}{2}}\sin^{-1}(\cot\tfrac{1}{2}x\tan\omega)\sin x\,dx$$

$$\therefore\int_\theta^{\frac{\pi}{2}}\{\sin^{-1}(\cot\tfrac{1}{2}x\tan\omega)+\sin^{-1}(\tan\tfrac{1}{2}x\tan\omega)\}\sin x\,dx$$

$$=\int_\theta^{\pi-\theta}\sin^{-1}(\cot\frac{x}{2}\tan\omega)\sin x\,dx$$

$$=\cos\theta\sin^{-1}(\tan\tfrac{1}{2}\theta\tan\omega)-2\sin\omega\sin^{-1}(\sin\tfrac{1}{2}\theta\sec\omega)-\tan^{-1}\{\cot\omega(\operatorname{cosec}^2\tfrac{1}{2}\theta-\sec^2\omega)^{-\frac{1}{2}}\}$$

$$+\cos\theta\sin^{-1}(\cot\tfrac{1}{2}\theta\tan\omega)+2\sin\omega\sin^{-1}(\cos\tfrac{1}{2}\theta\sec\omega)+\tan^{-1}\{\cot\omega(\sec^2\tfrac{1}{2}\theta-\sec^2\omega)^{-\frac{1}{2}}\}\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad(5)$$

Similarly

$$\int_\theta^{\frac{\pi}{2}}\{\sin^{-1}(\sin\tfrac{1}{2}x\sec\omega)+\sin^{-1}(\cos\tfrac{1}{2}x\sec\omega)\}\sin x\,dx$$

$$=\int_\theta^{\pi-\theta}\sin^{-1}(\sin\tfrac{1}{2}x\sec\omega)\sin x\,dx$$

and $$\int\sin^{-1}(\sin\tfrac{1}{2}x\sec\omega)\sin x\,dx$$

$$=\int 2\cos^2\omega\,\phi\sin 2\phi\,d\phi\quad\text{if}\quad\sin\tfrac{1}{2}x=\cos\omega\sin\phi$$

$$=\cos^2\omega\left(-\phi\cos 2\phi+\frac{\sin 2\phi}{2}\right)$$

$$=\sin\tfrac{1}{2}x(\cos^2\omega-\sin^2\tfrac{1}{2}x)^{\frac{1}{2}}-(\cos^2\omega-2\sin^2\tfrac{1}{2}x)\sin^{-1}(\sin\tfrac{1}{2}x\sec\omega)$$

$$\therefore \int_\theta^{\frac{\pi}{2}} \{\sin^{-1}(\sin \tfrac{1}{2}x \sec \omega) + \sin^{-1}(\cos \tfrac{1}{2}x \sec \omega)\} \sin x \, dx$$

$$= \cos \tfrac{1}{2}\theta(\cos^2 \omega - \cos^2 \tfrac{1}{2}\theta)^{\frac{1}{2}} - (\cos^2 \omega - 2 \cos^2 \tfrac{1}{2}\theta) \sin^{-1}(\cos \tfrac{1}{2}\theta \sec \omega) - \sin^2 \tfrac{1}{2}\theta(\cos^2 \omega - \sin^2 \tfrac{1}{2}\theta)^{\frac{1}{2}} + (\cos^2 \omega - 2 \sin^2 \tfrac{1}{2}\theta) \sin^{-1}(\sin \tfrac{1}{2}\theta \sec \omega) \quad \ldots \quad (6)$$

$$\tan \tfrac{1}{2} \{\sin^{-1}(\cot \tfrac{1}{2}x \tan \omega) + \sin^{-1}(\tan \tfrac{1}{2}x \tan \omega)\}$$

$$= \cot \tfrac{1}{2} \{\cos^{-1}(\cot \tfrac{1}{2}x \tan \omega) + \cos(\tan \tfrac{1}{2}x \tan \omega)\}$$

$$= \{1 - (1 - \tan \tfrac{1}{2}x \tan \omega)^{\frac{1}{2}}(1 + \tan \tfrac{1}{2}x \tan \omega)^{-\frac{1}{2}}(1 - \tan \omega \cot \tfrac{1}{2}x)^{\frac{1}{2}} (1 + \cot \tfrac{1}{2}x \tan \omega)^{-\frac{1}{2}}\} \{(1 - \tan \tfrac{1}{2}x \tan \omega)^{\frac{1}{2}} (1 + \tan \tfrac{1}{2}x \tan \omega)^{-\frac{1}{2}} + (1 - \cot \tfrac{1}{2}x \tan \omega)^{\frac{1}{2}} (1 + \cot \tfrac{1}{2}x \tan \omega)^{-\frac{1}{2}}\}^{-1}$$

$$= 2 \{(1 - \tan \tfrac{1}{2}x \tan \omega)^{\frac{1}{2}}(1 + \tan \tfrac{1}{2}x \tan \omega)^{-\frac{1}{2}}(1 + \cot \tfrac{1}{2}x \tan \omega)^{-1} - (1 - \cot \tfrac{1}{2}x \tan \omega)^{\frac{1}{2}}(1 + \cot \tfrac{1}{2}x \tan \omega)^{-\frac{1}{2}}(1 - \tan \tfrac{1}{2}x \tan \omega)^{-1}\} \{(1 - \tan \tfrac{1}{2}x \tan \omega)(1 + \tan \tfrac{1}{2}x \tan \omega)^{-1} - (1 - \cot \tfrac{1}{2}x \tan \omega) (1 + \cot \tfrac{1}{2}x \tan \omega)^{-1}\}^{-1}$$

$$= \cot \omega \left(\cot \frac{x}{2} - \tan \frac{x}{2}\right)^{-1} \{(1 - \tan^2 \tfrac{1}{2}x \tan^2 \omega)^{\frac{1}{2}} - (1 - \cot^2 \tfrac{1}{2}x \tan^2 \omega)^{\frac{1}{2}}\}$$

$$= 2^{-\frac{1}{2}} \operatorname{cosec} \omega \sec x \{\sin \frac{x}{2} (\cos x + \cos^2 \omega)^{\frac{1}{2}} - \cos \frac{x}{2} (\cos 2\omega - \cos x)^{\frac{1}{2}}\};$$

$$\therefore \int \tan^{1}[\sin \omega \tan \tfrac{1}{2} \{\sin^{-1}(\cot \tfrac{1}{2}x \tan \omega) + \sin^{-1}(\tan \tfrac{1}{2}x \tan \omega)\}] \sin x \, dx$$

$$= \int \tan^{-1}[2^{-\frac{1}{2}} \sec x \{\sin \tfrac{1}{2}x(\cos x + \cos 2\omega)^{\frac{1}{2}} - \cos \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}}\}] \sin x \, dx$$

$$= -\cos x \tan^{-1}[2^{-\frac{1}{2}} \sec x \{\sin \tfrac{1}{2}x(\cos x + \cos 2\omega)^{\frac{1}{2}} - \cos \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}}\}]$$

$$+ \int 2^{-\frac{1}{2}}[\sec x \{\tfrac{1}{2} \cos \tfrac{1}{2}x(\cos 2\omega + \cos x)^{\frac{1}{2}} - \tfrac{1}{2} \sin \tfrac{1}{2}x \sin x (\cos x + \cos 2\omega)^{-\frac{1}{2}} + \tfrac{1}{2} \sin \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}} - \tfrac{1}{2} \cos \tfrac{1}{2}x \sin x (\cos 2\omega - \cos x)^{\frac{1}{2}}\} + \sin x \sec^2 x \{\sin \tfrac{1}{2}x(\cos 2\omega + \cos x) - \cos \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}}\}][1 + \tfrac{1}{2} \sec^2 x \{\sin^2 \tfrac{1}{2}x (\cos x + \cos 2\omega) + \cos^2 \tfrac{1}{2}x(\cos 2\omega - \cos x) - \sin x (\cos^2 2\omega - \cos^2 x)^{\frac{1}{2}}]\cos x \, dx$$

The integral in this expression =

$$\int 2^{\frac{1}{2}} \cos x \{\cos \tfrac{1}{2}x(\cos 2\omega + \cos x)^{\frac{1}{2}} - \tfrac{1}{2} \cos x \cos \tfrac{1}{2}x(\cos 2\omega + \cos x)^{\frac{1}{2}} - \sin \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}} - \sin \tfrac{1}{2}x \cos x(\cos 2\omega - \cos x)^{\frac{1}{2}} - \tfrac{1}{2} \sin \tfrac{1}{2}x \sin x \cos x(\cos 2\omega + \cos x)^{-\frac{1}{2}} - \tfrac{1}{2} \cos \tfrac{1}{2}x \sin x \cos x (\cos 2\omega - \cos x)^{-\frac{1}{2}}\} \{\cos^2 x + \cos^2 2\omega - \sin x (\cos^2 2\omega - \cos^2 x)^{\frac{1}{2}}\}^{-1} dx$$

$$= 2^{\frac{1}{2}} \int \{\cos \tfrac{1}{2}x(\cos 2\omega + \cos x)^{\frac{1}{2}} - \cos x \cos \tfrac{1}{2}x \cos^2 \omega (\cos 2\omega + \cos x)^{-\frac{1}{2}} - \sin \tfrac{1}{2}x(\cos 2\omega - \cos x) - \cos x \sin \tfrac{1}{2}x \cos^2 \omega(\cos 2\omega - \cos x)^{-\frac{1}{2}}\} \{\cos \tfrac{1}{2}x(\cos 2\omega + \cos x)^{\frac{1}{2}} - \sin \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}}\}^{-2} dx$$

$$= 2^{\frac{1}{2}} \int \cos x \{\cos \tfrac{1}{2}x(\cos 2\omega + \cos x)^{\frac{1}{2}} - \sin \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}}\}^{-1} dx$$

$$- 2^{\frac{1}{2}} \cos^2 \omega \int \{\cos \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}} + \sin \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}}\} (\cos^2 2\omega - \cos^2 x)^{-\frac{1}{2}} \{\cos \tfrac{1}{2}x(\cos 2\omega + \cos x)^{\frac{1}{2}} - \sin \tfrac{1}{2}x (\cos 2\omega - \cos x)^{\frac{1}{2}}\}^{-2} \cos^2 x \, dx$$

$$= 2^{\frac{1}{2}} \int \{\cos \tfrac{1}{2}x(\cos 2\omega + \cos x)^{\frac{1}{2}} - \sin \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}}\}^{-1} \cos x \, dx$$

$$- 2^{-\frac{1}{2}} \int \{(\cos^2 2\omega - \cos^2 x)^{\frac{1}{2}} + \sin x \cos 2\omega\} (\cos^2 2\omega - \cos^2 x)^{-\frac{1}{2}} \{\cos \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}} - \sin \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}}\}^{-1} \cos x \, dx$$

$$= 2^{-\frac{1}{2}} \int \{\cos \tfrac{1}{2}x(\cos 2\omega + \cos x)^{\frac{1}{2}} - \sin \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}}\}^{-1} \cos x \, dx$$

$$- 2^{-\frac{1}{2}} \int \sin x \cos x \cos 2\omega(\cos^2 2\omega - \cos^2 x)^{-\frac{1}{2}} \{\cos \tfrac{1}{2}x (\cos 2\omega + \cos x)^{\frac{1}{2}} - \sin \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}}\}^{-1} \cos x \, dx$$

$$= 2^{-\frac{3}{2}} \sec^2 \omega \int \{\cos \tfrac{1}{2}x(\cos 2\omega + \cos x)^{\frac{1}{2}} + \sin \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}}\} dx$$

$$- 2^{-\frac{3}{2}} \cos 2\omega \sec^2 \omega \int \sin x(\cos^2 2\omega - \cos^2 x)^{-\frac{1}{2}} \{\cos \tfrac{1}{2}x (\cos 2\omega + \cos x)^{\frac{1}{2}} + \sin \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}}\} dx$$

$$= 2^{-\frac{1}{2}} \sec^2 \omega \{\sin \tfrac{1}{2}x(\cos 2\omega + \cos x)^{\frac{1}{2}} + \tfrac{1}{2} \int \sin \tfrac{1}{2}x \sin x (\cos 2\omega + \cos x)^{-\frac{1}{2}} dx - \cos \tfrac{1}{2}x \, (\cos 2\omega - \cos x)^{-\frac{1}{2}}$$
$$+ \tfrac{1}{2} \int \sin x \cos \tfrac{1}{2}x(\cos 2\omega - \cos x)^{-\frac{1}{2}} \, dx\} - 2^{-\frac{3}{2}} \cos 2\omega \sec^2 \omega \{ \int \sin x \cos \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}} dx + \int \sin x \sin \tfrac{1}{2}x(\cos 2\omega + \cos x)^{\frac{1}{2}} dx\}$$

$$= 2^{-\frac{1}{2}} \sec^2 \omega \{\sin \tfrac{1}{2}x(\cos 2\omega + \cos x)^{\frac{1}{2}} - \cos \tfrac{1}{2}x(\cos 2\omega - \cos x)^{\frac{1}{2}}\}$$
$$+ 2^{-\frac{1}{2}} \tan^2 \omega \{ \int \cos \tfrac{1}{2}x \sin x(\cos 2\omega - \cos x)^{-\frac{1}{2}} dx + \int \sin \tfrac{1}{2}x \sin x (\cos 2\omega + \cos x)^{\frac{1}{2}} dx\}$$

$$\int \cos \tfrac{1}{2}x \sin x(\cos 2\omega - \cos x)^{-\frac{1}{2}} dx = 2^{\frac{1}{2}} \int \cos^2 \tfrac{1}{2}x \sin \tfrac{1}{2}x(\cos^2 \omega - \cos^2 \tfrac{1}{2}x)^{-\frac{1}{2}} dx$$
$$= -2^{\frac{1}{2}} \cos^2 \omega \sin^{-1}(\cos \tfrac{1}{2}x \sec \omega) + 2^{\frac{1}{2}} \cos \tfrac{1}{2}x (\cos^2 \omega - \cos^2 \tfrac{1}{2}x)^{-\frac{1}{2}}.$$

Similarly

$$\int \sin \tfrac{1}{2}x \sin x(\cos 2\omega + \cos x)^{-\frac{1}{2}} dx = 2^{\frac{1}{2}} \cos^2 \omega \sin^{-1}(\sin \tfrac{1}{2}x \sec \omega) - 2^{\frac{1}{2}} \sin \tfrac{1}{2}x(\cos^2 \omega - \sin^2 \tfrac{1}{2}x)^{-\frac{1}{2}};$$

$$\therefore \int_{\theta}^{\frac{\pi}{2}} \tan^{-1}\left[\sin\omega \tan\tfrac{1}{2}\left\{\sin^{-1}(\cot\tfrac{1}{2}x \tan\omega) + \sin^{-1}(\tan\tfrac{1}{2}x\tan\omega)\right\}\right] \sin x\, dx$$

$$= \cos\theta \tan^{-1}\left[2^{-\frac{1}{2}}\sec\theta\left\{\sin\tfrac{1}{2}\theta(\cos\theta+\cos 2\omega)^{\frac{1}{2}} - \cos\tfrac{1}{2}\theta(\cos 2\omega - \cos\theta)^{\frac{1}{2}}\right\}\right] - 2^{-\frac{1}{2}}\left\{\sin\tfrac{1}{2}\theta(\cos 2\omega+\cos\theta)^{\frac{1}{2}} - \cos\tfrac{1}{2}\theta(\cos 2\omega - \cos\theta)^{\frac{1}{2}}\right\} + \sin^2\omega\left\{\sin^{-1}(\cos\tfrac{1}{2}\theta\sec\omega) - \sin^{-1}(\sin\tfrac{1}{2}\theta\sec\omega)\right\};$$

$\therefore$ from equations (4), (5), and (6), chance of conditions being fulfilled under this subcase,

$$= 2\pi^{-1}\Big\{\sin\omega\cos\tfrac{1}{2}\theta(\cos^2\omega - \cos^2\tfrac{1}{2}\theta)^{\frac{1}{2}} - \sin\omega(\cos^2\omega - 2\cos^2\tfrac{1}{2}\theta)\sin^{-1}(\cos\tfrac{1}{2}\theta\sec\omega) - \sin\omega\sin\tfrac{1}{2}\theta(\cos^2\omega - \sin^2\tfrac{1}{2}\theta)^{\frac{1}{2}} + \sin\omega(\cos^2\omega - 2\sin^2\tfrac{1}{2}\theta)\sin^{-1}(\sin\tfrac{1}{2}\theta\sec\omega) + \cos\theta\sin^{-1}(\tan\tfrac{1}{2}\theta\tan\omega) - 2\sin\omega\sin^{-1}(\sin\tfrac{1}{2}\theta\sec\omega) - \tan^{-1}\{\cot\omega(\operatorname{cosec}^2\tfrac{1}{2}\theta - \sec^2\omega)^{\frac{1}{2}}\} + \cos\theta\sin^{-1}(\cot\tfrac{1}{2}\theta\tan\omega) + 2\sin\omega\sin^{-1}(\cos\tfrac{1}{2}\theta\sec\omega) + \tan^{-1}\{\cot\omega(\sec^2\tfrac{1}{2}\theta - \sec^2\omega)^{\frac{1}{2}}\} - 4\sin\omega\cos\theta\tan^{-1}\left[2^{-\frac{1}{2}}\sec\theta\{\sin\tfrac{1}{2}\theta(\cos\theta+\cos 2\omega)^{\frac{1}{2}} - \cos\tfrac{1}{2}\theta(\cos 2\omega - \cos\theta)^{\frac{1}{2}}\}\right] + 2^{\frac{3}{2}}\sin\omega\{\sin\tfrac{1}{2}\theta(\cos 2\omega+\cos\theta)^{\frac{1}{2}} - \cos\tfrac{1}{2}\theta(\cos 2\omega - \cos\theta)^{\frac{1}{2}}\} - 4\sin^3\omega\{\sin^{-1}(\cos\tfrac{1}{2}\theta\sec\omega) - \sin^{-1}(\sin\tfrac{1}{2}\theta\sec\omega)\}\Big\} \quad \ldots \quad \ldots \quad (7)$$

9. $\therefore$ from equations (2), (3), and (7), total chance of conditions being fulfilled $= 2\tan^2\omega$

$$+ 2\pi^{-1}\sin\omega\left[\cos 2\omega\tan^{-1}\{\operatorname{cosec}\omega\cot\omega(\sin^2\tfrac{1}{2}\theta - \tan^2\omega)^{\frac{1}{2}}\} - \cos\theta\cos^{-1}\{\sin\omega\tan\omega(\sin^2\tfrac{1}{2}\theta - \sin^2\omega)^{-\frac{1}{2}}\} - 2\sin\omega\tan\omega(\sin^2\tfrac{1}{2}\theta - \tan^2\omega)^{\frac{1}{2}}\right]$$

$$+ 4\pi^{-1}\sin^2\tfrac{1}{2}\theta\sin^{-1}(\tan\omega\operatorname{cosec}\tfrac{1}{2}\theta) - 2\tan^2\omega + 4\pi^{-1}\tan\omega(\sin^2\tfrac{1}{2}\theta - \tan^2\omega)^{\frac{1}{2}}$$

$$+ 2\pi^{-1}\Big\{\sin\omega\cos\tfrac{1}{2}\theta(\cos^2\omega - \cos^2\tfrac{1}{2}\theta)^{\frac{1}{2}} - \sin\omega(\cos^2\omega - 2\cos^2\tfrac{1}{2}\theta)\sin^{-1}(\cos\tfrac{1}{2}\theta\sec\omega)$$

$$- \sin\omega\sin\tfrac{1}{2}\theta(\cos^2\omega - \sin^2\tfrac{1}{2}\theta)^{\frac{1}{2}} + \sin\omega(\cos^2\omega - 2\sin^2\tfrac{1}{2}\theta)\sin^{-1}(\sin\tfrac{1}{2}\theta\sec\omega)$$

$$+ \cos\theta\sin^{-1}(\tan\tfrac{1}{2}\theta\tan\omega) - 2\sin\omega\sin^{-1}(\sin\tfrac{1}{2}\theta\sec\omega) - \tan^{-1}\{\cot\omega(\operatorname{cosec}^2\tfrac{1}{2}\theta - \sec^2\omega)^{\frac{1}{2}}\}$$

$$+ \cos\theta\sin^{-1}(\cot\tfrac{1}{2}\theta\tan\omega) + 2\sin\omega\sin^{-1}(\cos\tfrac{1}{2}\theta\sec\omega) + \tan^{-1}\{\cot\omega(\sec^2\tfrac{1}{2}\theta - \sec^2\omega)^{\frac{1}{2}}\}$$

$$- 4\sin\omega\cos\theta\tan^{-1}\left[2^{-\frac{1}{2}}\sec\theta\{\sin\tfrac{1}{2}\theta(\cos\theta+\cos 2\omega)^{\frac{1}{2}} - \cos\tfrac{1}{2}\theta(\cos 2\omega - \cos\theta)^{\frac{1}{2}}\}\right] + 2^{\frac{3}{2}}\sin\omega\{\sin^{\frac{1}{2}}\theta(\cos 2\omega+\cos\theta)^{\frac{1}{2}} - \cos\tfrac{1}{2}\theta(\cos 2\omega - \cos\theta)^{\frac{1}{2}}\} - 4\sin^3\omega\{\sin^{-1}(\cos\tfrac{1}{2}\theta\sec\omega) - \sin^{-1}\sin\tfrac{1}{2}\theta\sec\omega)\}$$

T

$$= 4\pi^{-1} \sin^2 \frac{\theta}{2} \sin^{-1} (\tan \omega \operatorname{cosec} \tfrac{1}{2}\theta) + 2\pi^{-1} \sin \omega (\cos 2\omega - \cos \theta)$$
$$\cos^{-1} \{2^{\frac{1}{2}} \sin \omega \tan \omega (\cos 2\omega - \cos \theta)^{-\frac{1}{2}}\}$$
$$+ 2\pi^{-1} \sin \omega (\sin^2 \tfrac{1}{2}\theta - \tan^2 \omega)^{\frac{1}{2}} + 4\pi^{-1} \cos^2 \tfrac{1}{2} \theta \sin^{-1} (\tan \tfrac{1}{2}\theta \tan \omega)$$
$$- 4\pi^{-1} \sin^2 \tfrac{1}{2}\theta \sin^{-1} (\cot \tfrac{1}{2}\theta \tan \omega)$$
$$+ 2\pi^{-1} \sin \omega (3 \cos^2 \omega - 2 \sin^2 \tfrac{1}{2}\theta) \sin^{-1} (\cos \tfrac{1}{2}\theta \sec \omega) - 2\pi^{-1}$$
$$(3 \cos^2 \omega - 2 \cos^2 \tfrac{1}{2}\theta) \sin^{-1} (\sin \tfrac{1}{2}\theta \sec \omega)$$
$$+ 3\pi^{-1} 2^{\frac{1}{2}} \sin \omega \{\sin \tfrac{1}{2}\theta (\cos 2\omega + \cos \theta)^{\frac{1}{2}} - \cos \tfrac{1}{2} \theta (\cos 2\omega - \cos \theta)^{\frac{1}{2}}\}$$
$$- 8\pi^{-1} \cos \theta \sin \omega \tan^{-1} [2^{-\frac{1}{2}} \sec \theta \{\sin \tfrac{1}{2}\theta (\cos 2\omega + \cos \theta)^{\frac{1}{2}}$$
$$- \cos \tfrac{1}{2}\theta (\cos 2\omega - \cos \theta)^{\frac{1}{2}}\}].$$

10. We will now find the expansion of this in terms of ω as far as ω^3.

First $\theta = 2 \sin^{-1} [\frac{1}{8} \{8 - 5 \cos^2 \omega - \cos \omega (25 \cos^2 \omega - 16)^{\frac{1}{2}}\}]^{\frac{1}{2}}$ from the inequality (1)

$$= 2 \sin^{-1} \left\{ \tfrac{1}{8}\omega^2 \left(\frac{32}{3} - \frac{64\omega^2}{27} \right) \right\}^{\frac{1}{2}} q.p.$$
$$= 2 \sin^{-1} \{3^{-\frac{1}{2}} 2\omega (1 - \tfrac{1}{9}\omega^2)\} \, q.p.$$
$$= 4 \cdot 3^{-\frac{1}{2}} \omega (1 + \tfrac{1}{9}\omega^2) \, q.p.$$

So $\sin^{-1} (\tan \omega \operatorname{cosec} \tfrac{1}{2}\theta) = \sin^{-1} \{2 \cdot 3^{\frac{1}{2}} (1 + \tfrac{4}{9}\omega^2)\} \, q.p.$

$$= \tfrac{1}{3}\pi + 4 \cdot 3^{-\frac{3}{2}} \omega^2 \, q.p. ;$$

∴ the first term $= \tfrac{16}{9} \omega^2 \, q.p.$

Also $\cos 2\omega - \cos \theta = (1 - 2\omega^2) - \left(1 - \frac{\theta^2}{2}\right) q.p.$

$$= \tfrac{2}{3}\omega^2 \, q.p.$$

So that $\cos^{-1} \{2^{\frac{1}{2}} \sin \omega \tan \omega (\cos 2\omega - \cos \theta)^{-\frac{1}{2}}\}$

$$= \cos^{-1} (3^{\frac{1}{2}}\omega) \, q.p.$$
$$= \frac{\pi}{2} - \text{terms containing } \omega ;$$

∴ the second term $= \tfrac{2}{3}\omega^3 \, q.p.$

$$\sin^2 \tfrac{1}{2}\theta - \tan^2 \omega = \frac{\omega^2}{3} q.p. ;$$

∴ the third term $= 4 \cdot 3^{-\frac{1}{2}} \pi^{-1} \omega^2$

$$\sin^{-1} \left(\tan \frac{\theta}{2} \tan \omega \right) = \sin^{-1} (2 \cdot 3^{-\frac{1}{2}} \omega^2) ;$$
$$= 2 \cdot 3^{-\frac{1}{2}} \omega^2 \, q.p. ;$$

∴ the fourth term $= 8 \cdot 3^{-\frac{1}{2}} \cdot \pi^{-1} \omega^2$

$$\sin^{-1} (\cot \tfrac{1}{2}\theta \tan \omega) = \sin^{-1} \tfrac{1}{2} \{3^{\frac{1}{2}} (1 - \tfrac{2}{9}\omega^2)\} \, q.p.$$
$$= \frac{\pi}{3} - \text{terms in } \omega^2 ;$$

$\therefore$ the fifth term $= \frac{16}{9}\omega^2$ *q.p.*

$$3\cos^2\omega - 2\sin^2\tfrac{1}{2}\theta = 3 - \tfrac{17}{3}\omega^2 \text{ q.p.}$$

$$\sin^{-1}(\cos\tfrac{1}{2}\theta\sec\omega) = \sin^{-1}(1-\tfrac{1}{6}\omega^2) \text{ q.p.}$$

$$= \tfrac{1}{2}\pi - 3^{-\frac{1}{2}}\omega + \text{terms in } \omega^3;$$

$\therefore$ the sixth term $= 3\omega - 2 \cdot 3^{\frac{1}{2}} \cdot \pi^{-1}\omega^2 - \frac{37}{6}\omega^2$

$$3\cos^2\omega - 2\cos^2\tfrac{1}{2}\theta = 1 - \frac{\omega^2}{3} \text{ q.p.}$$

$$\sin^{-1}(\sin\tfrac{1}{2}\theta\sec\omega) = \sin^{-1}(2 \cdot 3^{-\frac{1}{2}}\omega) \text{ q.p.}$$

$$= 2 \cdot 3^{-\frac{1}{2}}\omega \text{ q.p.};$$

$\therefore$ the seventh term $= 4 \cdot 3^{-\frac{1}{2}}\pi^{-1}\omega^2$

$$\cos 2\omega + \cos\theta = 2 + \text{terms in } \omega^2;$$

$$\therefore \sin\tfrac{1}{2}\theta(\cos 2\omega + \cos\theta)^{\frac{1}{2}} - \cos\tfrac{1}{2}\theta(\cos 2\omega - \cos\theta)^{\frac{1}{2}}$$

$$= 3^{-\frac{1}{2}}\omega + \text{terms in } \omega^3;$$

$\therefore$ the eighth term $= 2 \cdot 3^{\frac{1}{2}} \cdot \pi\omega^2$ *q.p.*

and $\tan^{-1}\{\sin\frac{1}{2}\theta(\cos 2\omega + \cos\theta)^{\frac{1}{2}} - \cos\frac{1}{2}\theta(\cos 2\omega - \cos\theta)^{\frac{1}{2}}\}$

$$= \frac{\omega}{\sqrt{3}} + \text{terms in } \omega^3;$$

$\therefore$ the ninth term $= 8 \cdot 3^{-\frac{1}{2}} \cdot \pi^{-1}\omega^3$ *q.p.*

Collecting these terms and giving them their proper signs we get total chance $= 3\omega - \frac{11}{2}\omega^3$ *q.p.*

11. This is with the inradius expressed in radians.
Putting ω radians $= \delta$ degrees

we get chance $= \frac{1}{60}\pi\delta - \dfrac{11}{2 \times 180^3}\pi^3\delta^3$

$$= \cdot 0524\delta - \cdot 000029\,\delta^3 \text{ q.p.}$$

Putting $\delta = 4$ we get chance $= \cdot 202$ *q.p.*;

putting $\delta = 1$ we get chance $= \cdot 052$ *q.p.*

12. That the paths of three meteors should appear to meet in a point is not, however, a sufficient condition for their appearing to give a radiant. It is also necessary, (*a*) that the points at which they first appear should all be within 90° of one of the two points where their paths appear to meet, (*b*) that they should all be proceeding away from this point of meeting of their paths. Assuming that one direction is as probable as the other for each meteorite, the chance of the second of these conditions being fulfilled is clearly $\frac{1}{8}$.

13. To find the probability of the first being fulfilled let the zenith distance of P, one of the points of intersection of their

paths be z (fig. 3), then the area of that portion of the visible half of the celestial sphere which is within 90° of P is clearly $2\pi - 2z$.

The case now begins to present some difficulty, but, as a simple assumption, I will take a uniform distribution of points of intersection P, and also a uniform distribution of starting points. The former of these clearly agrees with my former assumption of a uniform distribution of tracks.

Then the chance of any one of the starting points of meteorites falling within 90° of P will be $\pi^{-1}(\pi - z)$, and the chance of the

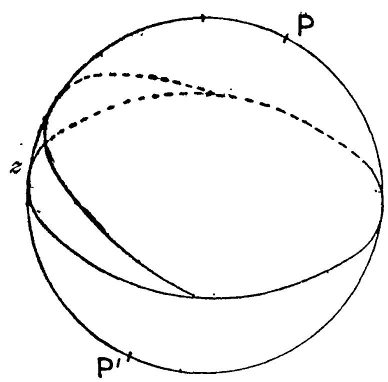

FIG. 3.

three falling within 90° of P is $\pi^{-3}(\pi - z)^3$, and the chance of the zenith distance of P lying between z and $z + dz = \sin z\, dz$.

∴ the total chance of the three points of first appearance lying within 90° of P, the meeting point of their paths above the horizon, $= \int_0^{\frac{\pi}{2}} \pi^{-3}(\pi - z)^3 dz$.

Besides this there is the possibility of their lying within 90° of P′, the point antipodal to P.

The chance of this is in a similar fashion $= \int_0^{\frac{\pi}{2}} \pi^{-3} z^3 dz$.

∴ the total chance of condition (a) being fulfilled

$$= \int_0^{\frac{\pi}{2}} \pi^{-3} z^3 \sin z\, dz + \int_0^{\frac{\pi}{2}} \pi^{-3}(\pi - z)^3 \sin z\, dz$$

$$= \int_0^{\pi} \pi^{-3} z^3 \sin z\, dz$$

$$= \pi^{-3}\Big[-z^3 \cos z + 3z^2 \sin z + 6z \cos z - 6 \sin z\Big]_0^{\pi}$$

$$= \pi^{-2}(\pi^2 - 6).$$

14. ∴ the chance of all three conditions being fulfilled and the three meteors appearing to indicate a radiant

$$= \cdot 202 \times \tfrac{1}{8} \times \pi^{-2}(\pi^2 - 6) = \cdot 0995 = \tfrac{1}{101}\ q.p. \text{ if } \delta = 4$$
$$= \cdot 052 \times \tfrac{1}{8} \times \pi^{-2}(\pi^2 - 6) = \cdot 0256 = \tfrac{1}{391}\ q.p. \text{ if } \delta = 1.$$

15. According to these results the process of determining a radiant from three meteors is in general sound, though the chance of at least one radiant of this sort being found among several meteors may be considerable. This and similar questions I may profitably leave to those of a greater practical acquaintance with the subject.

On the Variable Star Y Aurigæ (*Ch.* 1929).
By A. Stanley Williams.

$$\text{R.A.} = 5^{\text{h}}\ 18^{\text{m}}\ 20^{\text{s}}, \text{ Decl.} = +42^{\circ}\ 18'.5\ (1855)$$
$$\text{,,} = 5\ 21\ 32, \text{ ,,} = +42\ 21'.1\ (1900)$$

When the unknown period of a variable star is longer than a few days there is usually little difficulty in ascertaining its real length, since both this and the form of the light curve can generally be derived directly from the observations made night by night. There is likewise not usually any very great difficulty when the period is much less than a day, for then the variation is so rapid that the observations of a few consecutive hours are sufficient to indicate the correct period. It is when the period falls between about one day and four or five days that the most difficulty arises, and the difficulty is intensified when the increase in brightness is much more rapid than the decrease, since in such cases a marked change in the brightness of the star may be noticed in a few hours, giving rise to the impression that the period is a much shorter one than it really is. The variable *Y Aurigæ* belongs to this last-mentioned class. In the *Astronomische Nachrichten*, No. 3708, the period of this star was stated by the writer to be $0^{\text{d}}.7925$, this period satisfying all the data at the time available, whilst some observations made during the increase seemed to show that the period must be a very short one, there being a considerable increase in brightness in the course of three or four hours. But recently, upon reducing the observations of this star made here during the past four years, it was found that this very short period failed to satisfy them for any very long interval of time; nor could any variation of the same very short period be made to do so. On the other hand, a period of $3^{\text{d}}.862$, or thereabouts, seemed to satisfactorily represent all the observations. The present paper contains a discussion of the observations made here during the last four years, and in order that the whole of the evidence as to the real period and light curve may be accessible, the actual observations have been stated in full.

The observations in question were all made with a power of 75 on a $2\frac{3}{4}$-inch refractor, and in order to eliminate the effects